

9/12 Recitation

1. Let's use Julia to help us find an LU decomposition for $A = \begin{bmatrix} 3 & -1 & 0 \\ -9 & 5 & 3 \\ 3 & 5 & 5 \end{bmatrix}$ and then use

it to solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$.

- (a) Work through the first section of the **Julia6LUFactorization** notebook to find an LU factorization of A and to use it to solve the system.

See solution in the notebook.

- (b) Use the `lu(A)` function defined in the second section in the Julia notebook to let Julia find an LU factorization of A (it really provides a $PA = LU$ factorization).

Use the notebook.

- (c) How do the factorizations compare? Are they the same? If they are different, what did Julia do differently? What's going on here?

In general, because a matrix can be reduced to different upper triangular matrices, LU decompositions are not unique. Hence A (or PA) can be written as a product of different pairs of lower triangular and upper triangular matrices. For example

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -\frac{3}{2} \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

- (a) Find the $PA = LU$ factorization.

We start by performing row operations to put A in upper triangular form.

$$\begin{aligned}
\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} &\xrightarrow{\text{Swap R1 and R2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\
&\xrightarrow{\text{Replace R3 by R3-2R1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \\
&\xrightarrow{\text{Replace R3 by R3-3R2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U
\end{aligned}$$

These three row operations correspond to the elimination matrices

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

and $U = E_{32}E_{21}P_{12}A$. Multiplying both sides by $(E_{32}E_{21})^{-1}$ yields $P_{12}A = E_{21}^{-1}E_{32}^{-1}U$. Setting $P = P_{12}$ and $L = E_{21}^{-1}E_{32}^{-1}$ gives $PA = LU$.

$$\text{Now } L = E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Thus we have:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) Verify your factorization is correct by computing both PA and LU .

$$\begin{aligned}
PA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\
LU &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}
\end{aligned}$$

- What row operation does multiplication on the left by E_{ij} encode? What row operation does multiplication on the left by E_{ij}^{-1} encode? Combine these two operations to determine what effect multiplication on the left by $E_{ij}E_{ij}^{-1}$ has. How does this compare to the fact that $E_{ij}E_{ij}^{-1} = I$?

Multiplication by E_{ij} with multiplier l_{ij} replaces Row j by (Row $j - l_{ij}$ Row i). E_{ij}^{-1} is the same as E_{ij} , except with the multiplier negated; hence multiplication by E_{ij}^{-1} replaces Row j by (Row $j + l_{ij}$ Row i). Multiplication by $E_{ij}E_{ij}^{-1}$ corresponds to first replacing (Row j) by (Row $j + l_{ij}$ Row i) and then replacing (new Row j)—which is (Row $j + l_{ij}$ Row i)—by (new Row $j - l_{ij}$ Row i)—which is ((Row $j + l_{ij}$ Row i) - l_{ij} Row i) = (Row j). Hence multiplication by $E_{ij}E_{ij}^{-1}$ does not change the matrix; this makes sense, since $E_{ij}E_{ij}^{-1} = I$ and multiplication by I doesn't change the matrix.

4. Multiplication on the left of a matrix A by a permutation matrix swaps the rows of A . What does multiplication on the right by a permutation matrix do? Use Julia to define some 3×3 permutation matrices and a fixed matrix A . What does multiplication by the permutation matrices on the right do?

Multiplication on the left of A by P_{ij} swaps the i th and j th rows of A . Multiplication on the right swaps the i th and j th columns of A .

5. Write down a 3×3 lower triangular matrix L and a 3×3 upper triangular matrix U . Use Julia to compute the product $A = LU$ and use Julia to check whether A is invertible by using the `inv(A)` command. Repeat this with a new pair of lower triangular and upper triangular matrices. And a new pair. And a new pair. Keep repeating until you have found many examples of A being invertible and many examples of A being noninvertible. Try to find a pattern and come up with conditions on the entries of L and U that lead A to be invertible. Once you have a conjecture, prove it!

Conjecture A is invertible if L and U do not have 0s on their diagonal.

First, if U has no zeros on its diagonal, then it has n pivots and hence is invertible. Similarly, if L has no zeros on its diagonal, then by performing Gaussian elimination, we can reduce L to an upper triangular matrix (really, a diagonal matrix) with no zeros on its diagonal; hence L also has n pivots and it thus invertible. It now follows that A is invertible with inverse $U^{-1}L^{-1}$.